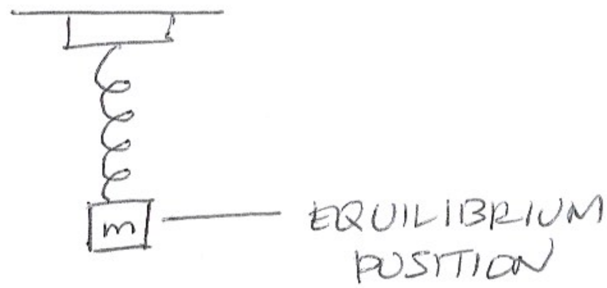
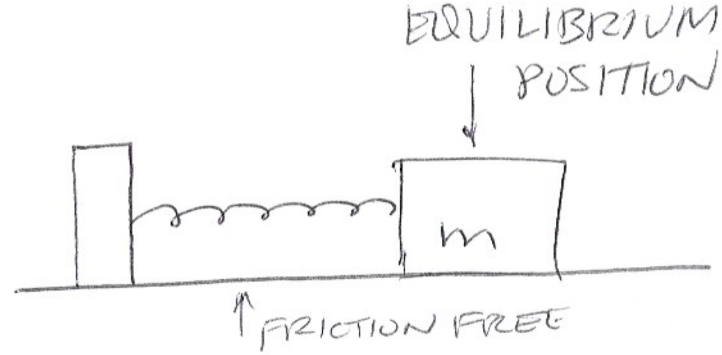
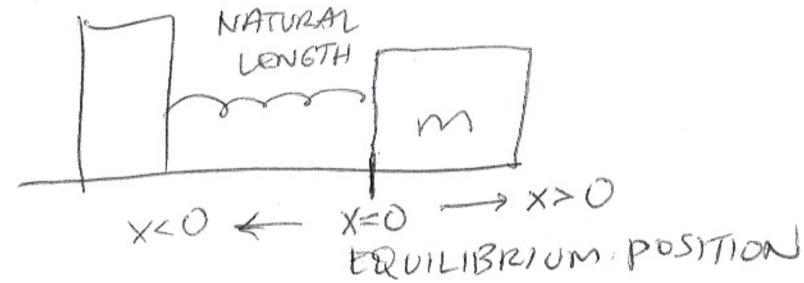
MASS SPRING SYSTEMS







NEWTON'S LAW

$$\vec{F} = m\vec{a}$$

$$\vec{F} = \sum_i \vec{F}_i$$

HOOKE'S LAW

$F_{SPR} \propto$ LENGTH SPRING IS STRETCHED/COMPRESSED

$F_{SPR} = -kx,\ k>0$

IF $x>0$, SPRING STRETCHED

FORCE OF SPRING F TO LEFT

$F<0$

IF $x<0$, SPRING COMPRESSED

FORCE OF SPRING F TO RIGHT

$F>0$

$$ma = -kx$$

$$mx'' = -kx$$

$$mx'' + kx = 0$$

$mx'' + kx = 0 \qquad m, k > 0$

$mr^2 + k = 0$

$r = \pm\sqrt{\frac{k}{m}}\, i \longrightarrow x = c_1 \cos\sqrt{\frac{k}{m}}\,t + c_2 \sin\sqrt{\frac{k}{m}}\,t$

$x = A\cos\left(\sqrt{\frac{k}{m}}\,t + \phi\right)$ FROM MATH 42

$\text{PERIOD} = \dfrac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}}$

AS m INCREASES, PERIOD INCREASES CONFIRMED $\frac{m}{k}\uparrow$ $\quad 2\pi\sqrt{\frac{m}{k}}\uparrow$

AS k INCREASES, PERIOD DECREASES CONFIRMED $\frac{m}{k}\downarrow$ $\quad 2\pi\sqrt{\frac{m}{k}}\downarrow$
(STIFFER SPRING)

UNREALISTIC BECAUSE
MASS OSCILLATES FOREVER
BETWEEN $-A$ AND A

F_{FLUID} OR $F_{AIR} = -bx'$, $b>0$

$mx'' = -bx' - kx$

$mx'' + bx' + kx = 0 \quad m, b, k > 0$

$mr^2 + br + k = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

① $r = r_1, r_2 \in \mathbb{R}$ AND $r_1 \neq r_2$

② $r = r_1, r_1 \in \mathbb{R}$

③ $r = \alpha \pm \beta i \quad \alpha, \beta \in \mathbb{R}$

① $r = -r_1, -r_2 \in \mathbb{R}$, $r_1 \neq r_2$ $\quad r_1, r_2 > 0$

$x = c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$ (ASSUME $r_1 > r_2$)

SANITY CHECK: AS $t \to \infty$, $x \to c_1 \cdot 0 + c_2 \cdot 0 = 0$

$b^2 > b^2 - 4mk > 0$

$\sqrt{b^2} > \sqrt{b^2 - 4mk}$

$b > \sqrt{b^2 - 4mk} \longrightarrow -b < -\sqrt{b^2 - 4mk}$

$-b + \sqrt{b^2 - 4mk} < 0$

$-b \pm \sqrt{b^2 - 4mk}$

$-b + \sqrt{b^2 - 4mk} < 0 \qquad -b - \sqrt{b^2 - 4mk} < 0$

$b^2 - 4mk > 0$

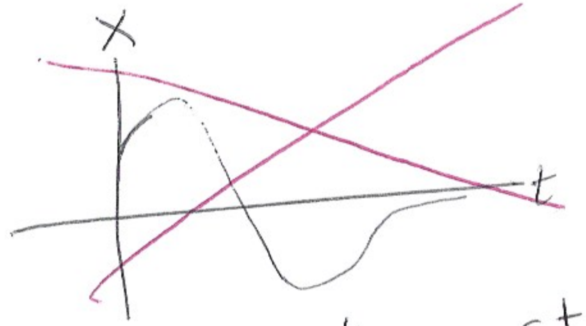
$b^2 > 4mk$

$b > 2\sqrt{mk}$

$\uparrow$

STRONG FLUID RESISTANCE $\longrightarrow$ FINITE OSCILLATION ?

OVERDAMPED



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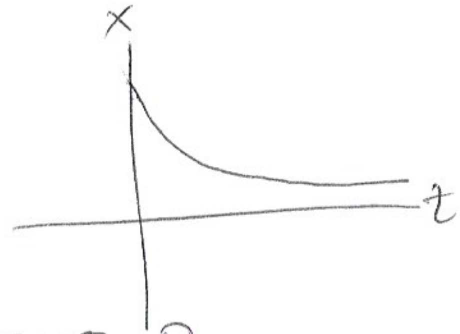

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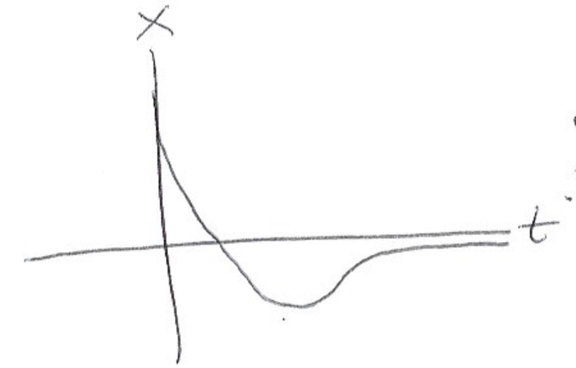
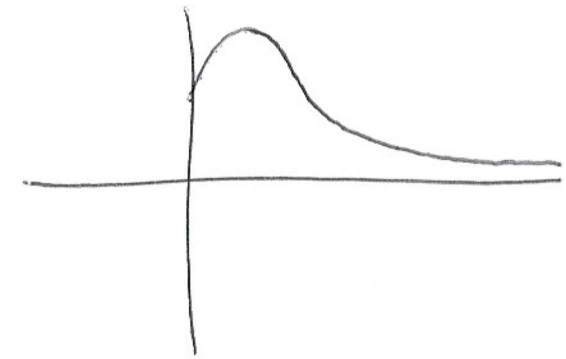




$$x = c_1 e^{-r_1 t} + c_2 e^{-r_2 t} \qquad r_1 > r_2 > 0$$

$$x' = -c_1 r_1 e^{-r_1 t} - c_2 r_2 e^{-r_2 t} = 0$$

$$e^{r_1 t}(-c_1 r_1 e^{-r_1 t}) = (c_2 r_2 e^{-r_2 t}) e^{r_1 t} \qquad e^{r_1 t} \neq 0$$

$$-c_1 r_1 = c_2 r_2 e^{(r_1 - r_2)t}$$

$$\frac{1}{r_1 - r_2} \ln\left(\frac{-c_1 r_1}{c_2 r_2}\right) = t$$

IF $c_1, c_2 > 0$, $t \notin \mathbb{R} \longrightarrow$ NO LOCAL EXTREMA

IF $c_1, c_2 < 0$, $t \notin \mathbb{R} \longrightarrow$ "

IF $c_1 > 0$, $c_2 < 0$
OR $c_1 < 0$, $c_2 > 0$ $\qquad -\dfrac{c_1 r_1}{c_2 r_2} > 0$

IF $0 < \dfrac{-c_1 r_1}{c_2 r_2} < 1$, $t < 0 \longrightarrow$ NO LOCAL EXTREMA

IF $\dfrac{-c_1 r_1}{c_2 r_2} > 1 \longrightarrow t$ HAS 1 REAL POSITIVE VALUE

1 LOCAL EXTREMA

(2) $r = -r_1, -r_1 \in \mathbb{R}$, $r_1 > 0$

$m, k, b > 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \rightarrow b^2 - 4mk = 0$$

$$b = 2\sqrt{mk}$$

$$x = c_1 e^{-r_1 t} + c_2 t e^{-r_1 t}$$

$$r = \frac{-b}{2m}, \frac{-b}{2m} < 0$$

As $t \to \infty$, $e^{-r_1 t} \to 0$

$te^{-r_1 t} \to 0$ (L'H)

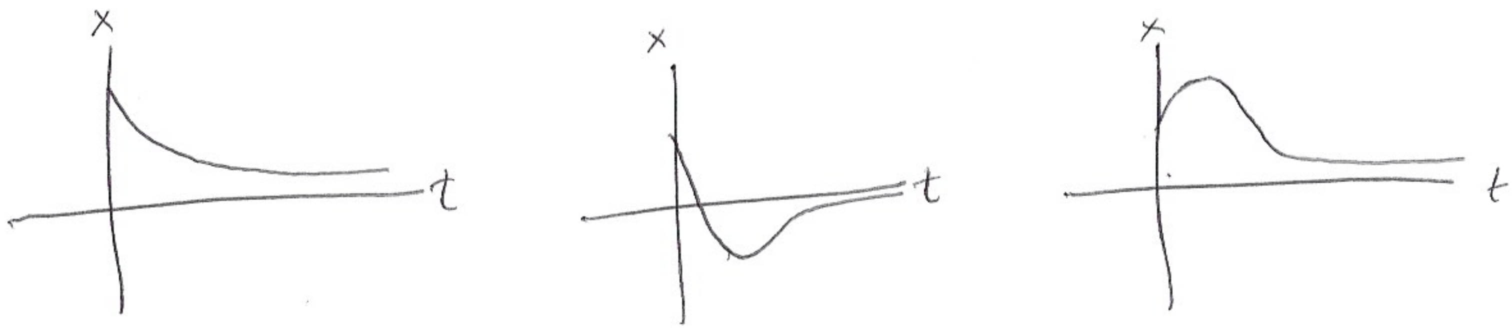
so, $x \to 0$

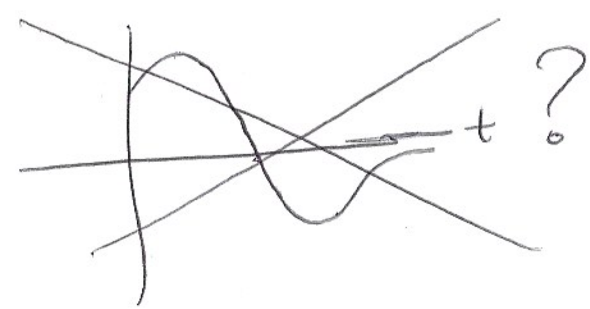
$x = (c_1 + c_2 t)e^{-r_1 t} = 0 \rightarrow$ PASS THROUGH EQUILIBRIUM

$c_1 + c_2 t = 0$

IF $t = \frac{-c_1}{c_2} < 0 \longrightarrow$ NEVER PASS THROUGH EQUILIBRIUM

IF $t = \frac{-c_1}{c_2} > 0 \longrightarrow$ PASS THROUGH EQ @ POSITIVE t VALUE ONCE





LOCAL EXTREMA

$$x = (c_1 + c_2 t)e^{-r_1 t}$$

$$x' = c_2 e^{-r_1 t} + -r_1 (c_1 + c_2 t)e^{-r_1 t}$$

$$= (c_2 - r_1 c_1 - r_1 c_2 t)e^{-r_1 t} = 0$$

$$c_2 - r_1 c_1 - r_1 c_2 t = 0$$

$$c_2 - r_1 c_1 = r_1 c_2 t$$

$$t = \frac{c_2 - r_1 c_1}{r_1 c_2} = \frac{-r_1 c_1}{r_1 c_2} + \frac{1}{r_1}$$

$$= \frac{-c_1}{c_2} + \frac{1}{r_1}$$

$\uparrow$

ONLY 1 t @ WHICH MASS PASSES THROUGH EQUILIBRIUM

ONLY 1 t @ WHICH MASS REACHES LOCAL EXTREMA

(3) $r = -\alpha \pm \beta i$

$$x = c_1 e^{-\alpha t}\cos\beta t + c_2 e^{-\alpha t}\sin\beta t$$

As $t \to \infty$, $x \to c_1 \cdot 0 + c_2 \cdot 0 = 0$

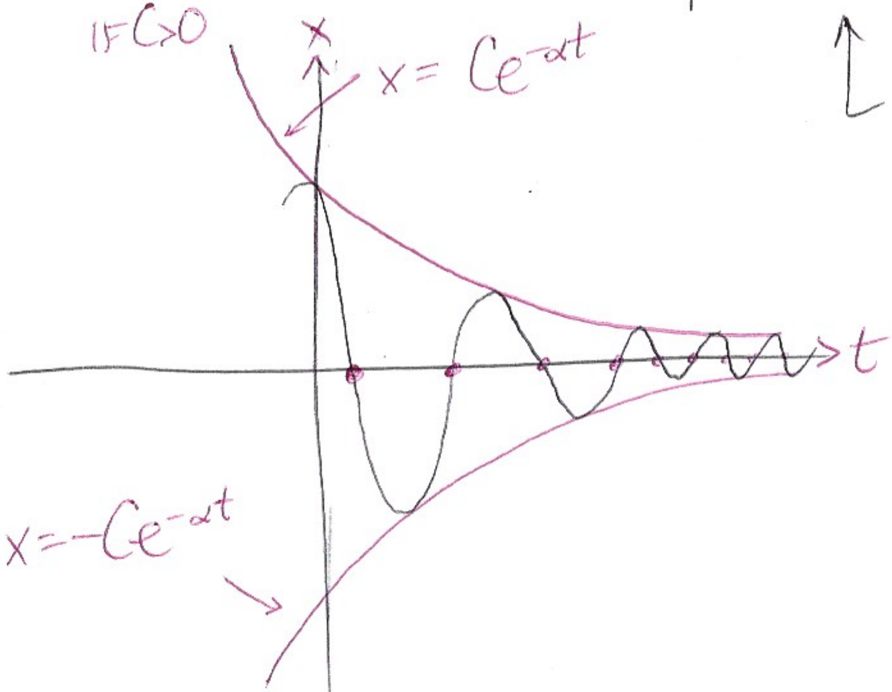
(SQUEEZE TH'M)

$$x = e^{-\alpha t}\left(c_1 \cos\beta t + c_2 \sin\beta t\right)$$

$$x = e^{-\alpha t}\left(C\cos(\beta t + \phi)\right)$$

$$x = Ce^{-\alpha t}\cos(\beta t + \phi)$$

NOT PERIODIC



$mr^2 + br + k$

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

$b^2 - 4m$

$\downarrow$

$b < 2\sqrt{}$

$$r = \frac{-b \pm \sqrt{4mk - b^2}\, i}{2m}$$

$$r = \underbrace{\frac{-b}{2m}}_{-\alpha} \pm \underbrace{\frac{\sqrt{4mk - b^2}}{2m}}_{\beta} i$$

$\alpha = \frac{b}{2m}$

$$\cos\theta = \sin\left(\tfrac{\pi}{2} - \theta\right)$$

$$\cos(\beta t + \phi) = \sin\left(\tfrac{\pi}{2} - (\beta t + \phi)\right)$$

$$= \sin\left(-(\beta t + \phi - \tfrac{\pi}{2})\right)$$

$$= -\sin\left(\beta t + \underline{\phi - \tfrac{\pi}{2}}\right)$$

$$x = Ce^{-\alpha t}\cos(\beta t + \phi) = 0$$

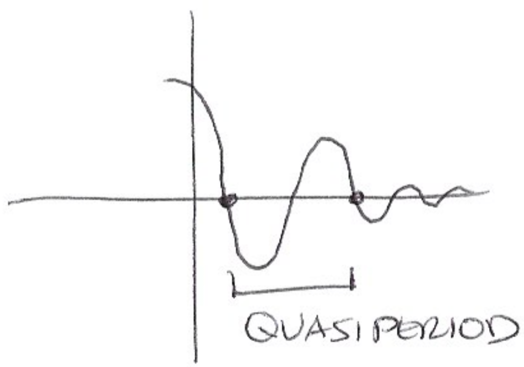
$$\cos(\beta t + \phi) = 0$$

$$\beta t + \phi = \frac{\pi}{2} + n\pi$$

$$t = \frac{\frac{\pi}{2} + n\pi - \phi}{\beta} = \frac{\frac{\pi}{2} - \phi}{\beta} + n\left(\frac{\pi}{\beta}\right)$$

$$1 \text{ QUASIPERIOD} = \frac{2\pi}{\beta} = \frac{2\pi}{\frac{\sqrt{4mk-b^2}}{2m}} = \frac{4\pi m}{\sqrt{4mk-b^2}}$$

$$= \frac{4\pi}{\sqrt{\frac{4mk-b^2}{m^2}}}$$



AS m INCREASES, QUASIPERIOD INCREASES

BUT NOT PAST $2\sqrt{mk}$ (IE. $b < 2\sqrt{mk}$)

AS b INCREASES, QUASIPERIOD INCREASES CONFIRMED $b^2 \uparrow$ $\frac{4\pi m}{}$

(HIGHER FLUID RESISTANCE) $4mk - b^2 \downarrow$ $\sqrt{4mk-}$

AS k INCREASES, QUASIPERIOD DECREASES CONFIRMED $4mk \uparrow$ $\frac{4\pi m}{\sqrt{4mk-b^2}}$

(STIFFER SPRING) $4mk - b^2 \uparrow$